Universal formula for robust stabilization of affine nonlinear multistable systems

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***Abstract*—In this paper, the problem of robust stabilization of affine nonlinear multistable systems with respect to disturbance inputs is studied. The results are obtained using the framework of input-to-state stability (ISS) and integral input-to-state stability (iISS) for systems with multiple invariant sets. The notions of ISS and iISS control Lyapunov functions as well as the small control property are extended within the multistability framework. It is verified that the universal control formula can be applied to yield the ISS (iISS) property to the closed-loop system. The efficiency of the proposed control Lyapunov function in the multistable sense is illustrated in two academic examples.**

I. Introduction

Due to its importance for several scientific disciplines ranging from mechanics and electronics [19], [13] to biology [26], [20], [24] and neurosciences [25], the analysis of stability and robustness properties of multistable systems have become increasingly attractive from the perspective of systems and control theory. Multistable systems include bistable systems (with at least two stable equilibria) [34], [7], almost globally stable systems (with only one attracting invariant set) [2], and nonlinear systems with generic invariant sets [4], [9], [14], [17], [18], [27], [32], [33]. In [12], it was proposed a global asymptotic stability notion as well as the necessary and sufficient Lyapunov characterization for multistable systems, having as the object of investigation all compact invariant solutions of the system (including locally stable and unstable ones).

By virtue of nontrivial relationships between different regions that compose its state space, and the complex intertwined boundaries between then, multistable systems are extremely sensitive to initial conditions and perturbations. The characterization introduced in [12] has been also proved to be useful in robustness analysis with respect to external disturbances. In fact, it was made clear in [3] that the most natural way to solve this problem is to relax the Lyapunov stability requirement on relatively mild additional assumptions on the decomposition of invariant sets. This intuitive path has led to a new line of research which starts from the characterizations of input-to-state stability (ISS) for this class of systems in terms of usual Lyapunov dissipation inequalities, generalizing the classical ISS theory [30], [31], [8], [3], [15]. In its turn, integral input-to-state stability (iISS) characterization, which is weaker than the classical one given in [29], [21], [5], was extended in [16] for systems with multiple invariant sets.

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Once ISS and iISS characterizations in the multistable sense are already available, the research interest on the problem of designing robust stabilizing control laws in this framework naturally increases. In this setting, such a problem consists in finding state feedback control laws that make the closed loop system ISS or iISS stable with respect to a family of finite disjoint compact invariant sets against external disturbances effects. In the classical approach, most of this activity is centered around the control Lyapunov function (CLF) theory [6], [28], [30], [11], [10]. In [6], it was shown that the existence of a CLF leads to an explicit formulation for stabilizing control laws. Similar results were proven in [22] for the ISS (iISS) case resulting in appropriated universal formulation for the assignment of a ISS (iISS) CLF rendering ISS (iISS) properties to the closed-loop system. In the present work, we are interested in the robust stabilization of multistable affine nonlinear systems with respect to disturbance inputs. Our approach is based on the theory developed in [12], [3], [16] and it aims to find conditions of CLF existence in the context of systems with multiple invariant sets (compact and maybe disconnected) and show how such a control can be explicitly designed for robust stabilization in ISS (iISS) sense.

The outline of this work is as follows. The main definitions and problem statement are given in Section II, while the obtained results are presented in Section III. In section IV are presented two examples of application of the proposed CLF approach. Final remarks and discussion are summarized in Section V.

II. Definitions and Problem Statement

Let $\mathcal{M}$ be a n-dimensional smooth manifold without boundary, equipped with a metric $\delta : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$, $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$. Consider a nonlinear model of dynamical systems evolving on this manifold:

$$\dot{x}(t) = f(x(t), u(t)), \quad \forall\, t \in \mathbb{R}_+, \tag{1}$$

$$y(t) = h(x(t)), \tag{2}$$

where $x(t) \in \mathcal{M}$ is the state vector, $u(t) \in \mathbf{U} \subseteq \mathbb{R}^m$ is the input vector, u is an element of $\mathcal{U}$, the set of admissible controls $\mathbb{R}_+ \to \mathbf{U}$ (locally essentially bounded and measurable signals), and $y(t) \in \mathcal{Y} \subseteq \mathbb{R}^p$ is the output vector. Let

$f : \mathcal{M}\times\mathbf{U} \to T_x\mathcal{M}$ be a locally Lipschitz continuous function on $\mathcal{M}$ (here $T_x\mathcal{M}$ denotes the tangent space of $\mathcal{M}$ at x), and assume that $h : \mathcal{M} \to \mathcal{Y}$ is continuously differentiable, $h(0) = 0$ and $f(0, 0) = 0$.

Denote by $x(t, x_0; u)$ the uniquely defined solution of (1) at time $t \geq 0$ such that $x(0) = x_0$ under the input $u \in \mathcal{U}$. For the unperturbed system, *i.e*. the system (1) with $u \equiv 0$, we have:

$$\dot{x}(t) = f(x(t), 0), \quad t \geq 0, \tag{3}$$

and we say that $S \subset \mathcal{M}$ is *invariant* if for all $x_0 \in S$, $x(t, x_0; 0) \in S$ for all $t \in \mathbb{R}$.

For a set $S \subset \mathcal{M}$ and $x \in \mathcal{M}$ define the corresponding distance as

$$|x|_S = \inf_{a\in S} \delta(x, a).$$

For a measurable function $g : \mathbb{R}_+ \to \mathbb{R}^m$, define its L_∞-norm as

$$||g||_\infty = \operatorname*{ess\,sup}_{t\geq 0} |g(t)|.$$

A. Decomposition of a compact invariant set

Let $\Lambda \subset \mathcal{M}$ be a compact invariant set for the unperturbed system (3). To characterize the evolution of this system along $\mathcal{M}$, it is useful to decompose Λ and explicitly determine the existence of solutions traveling between different components of its decomposition.

Definition 1. [23] A decomposition of Λ is a finite, disjoint family of compact invariant sets $\Lambda_1, \cdots , \Lambda_k$ such that $\Lambda = \bigcup_{i=1}^k \Lambda_i$.

For an invariant set Λ, its attracting and repulsing subsets can be defined, respectively, as follows:

$$\mathfrak{A}(\Lambda) = \{x_0 \in \mathcal{M} : |x(t, x_0; 0)|_\Lambda \to 0 \text{ as } t \to +\infty\},$$
$$\mathfrak{R}(\Lambda) = \{x_0 \in \mathcal{M} : |x(t, x_0; 0)|_\Lambda \to 0 \text{ as } t \to -\infty\}.$$

Based on these definitions, we can define a relation between two invariant sets $\mathcal{W} \subset \mathcal{M}$ and $\mathcal{D} \subset \mathcal{M}$ by $\mathcal{W} \prec \mathcal{D}$ if $\mathfrak{A}(\mathcal{W}) \cap \mathfrak{R}(\mathcal{D}) \neq \emptyset$. This relation implies that there is a solution connecting the set $\mathcal{D}$ with the set $\mathcal{W}$. A collection of r disjoint sets that can be reached from one to another in a loop by a suitable concatenation of systems solutions is called r-cycle.

Definition 2. [23] Let $\Lambda_1, \cdots , \Lambda_k$ be a decomposition of Λ.

1) An r-cycle ($r \geq 2$) is an ordered r-tuple of distinct indices $i_1, \cdots , i_r$ such that $\Lambda_{i_1} \prec \cdots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.

2) A 1-cycle is an index i such that $(\mathfrak{R}(\Lambda_i) \cap \mathfrak{A}(\Lambda_i)) \setminus \Lambda_i \neq \emptyset$.

3) A filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \leq j$.

However, what qualifies a decomposition for its treatment by means of Lyapunov-like analytical tools is the absence of these cycles. Therefore, we will next consider the following assumption:

Assumption 1. *[3] A compact invariant set $\mathcal{W}$ containing all α- and ω-limit sets of the unperturbed system* (3)*, admits a finite decomposition without cycles: $\mathcal{W} = \bigcup_{i=1}^k \mathcal{W}_i$ for some non-empty disjoint compact sets $\mathcal{W}_i$ which forms a filtration ordering of $\mathcal{W}$, as detailed in Definition 2.*

B. Robust stability notions for a decomposable compact invariant set $\mathcal{W}$

In this subsection, we list several ISS and iISS stability properties for (1) with respect to $\mathcal{W}$ satisfying Assumption 1. Most of these properties are direct extensions of the classical ISS and iISS notions introduced in [30], [31], [21], [5]. The definitions of function classes $\mathcal{K}$ and $\mathcal{K}_\infty$ can be found in [8]. A function $V : \mathcal{M} \to \mathbb{R}_+$ is called *positive definite* if it vanishes only at the origin, and *proper unbounded* if $V(x) \to +\infty$ for $|x|_\mathcal{W} \to +\infty$. The Lie derivative of a continuously differentiable function V along a vector field $f : \mathcal{M} \to \mathbb{R}^n$ is denoted by:

$$DV(x)f(x, u) = \frac{\partial V(x)}{\partial x} f(x, u).$$

Definition 3. [3], [16] The system (1) has the *practical asymptotic gain (pAG)* property if there exist $\eta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x_0 \in \mathcal{M}$ and all $u \in \mathcal{U}$, the solutions are defined for all $t \geq 0$ and the following holds:

$$\limsup_{t\to+\infty} |x(t, x_0; u)|_\mathcal{W} \leq \eta(||u||_\infty) + q. \tag{4}$$

If $q = 0$, then we say that the *asymptotic gain (AG)* property holds. Moreover, if (4) is satisfied for $q = 0$ for the system (3) only, then we will say that (1) has the *zero-global attraction (0-GATT)* property with respect to a compact invariant set $\mathcal{W}$.

Definition 4. [3] The system (1) has the *limit property (LIM)* with respect to $\mathcal{W}$ if there exists $\mu \in \mathcal{K}_\infty$ such that for all $x_0 \in \mathcal{M}$ and all $u \in \mathcal{U}$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\inf_{t\geq 0} |x(t, x_0; u)|_\mathcal{W} \leq \mu(||u||_\infty).$$

Definition 5. [3] The system (1) has the *practical global stability (pGS)* property with respect to $\mathcal{W}$ if there exists $\beta \in \mathcal{K}_\infty$ and $c \geq 0$ such that for all $x_0 \in \mathcal{M}$ and all $u \in \mathcal{U}$, the following holds for all $t \geq 0$:

$$|x(t, x_0; u)|_\mathcal{W} \leq \beta \left(\max\{|x_0|_\mathcal{W} + c, ||u||_\infty\}\right).$$

Definition 6. [3], [16] A $\mathcal{C}^1$ function $V : \mathcal{M} \to \mathbb{R}$ is a *practical ISS Lyapunov function* for (1) if there exist $\mathcal{K}_\infty$ functions α_1, α_2, α_3 and γ, and $q \geq 0$, $c \geq 0$ such that

$$\alpha_1(|x|_\mathcal{W}) \leq V(x) \leq \alpha_2(|x|_\mathcal{W}) + c \tag{5}$$

and the following dissipation inequality holds for all $(x, u) \in \mathcal{M} \times \mathcal{U}$:

$$DV(x)f(x, u) \leq -\alpha_3(|x|_\mathcal{W}) + \gamma(|u|) + q. \tag{6}$$

If (6) holds for $q = 0$, then V is said to be an *ISS Lyapunov function*. If (6) holds for $q = 0$ and a positive definite function $\alpha_3 : \mathbb{R}_+ \to \mathbb{R}_+$, then V is said to be an *iISS Lyapunov function*.

Definition 7. [16] The system (1) has the *uniform bounded-energy bounded-state (UBEBS)* property if for some $\alpha, \gamma, \sigma \in \mathcal{K}_\infty$ and some positive constant c, the following estimate holds for all $t \geq 0$, all $x_0 \in \mathcal{M}$ and all $u \in \mathcal{U}$:

$$\alpha(|x(t, x_0; u)|_{\mathcal{W}}) \leq \gamma(|x_0|_{\mathcal{W}}) + \int_0^t \sigma(|u(\tau)|)d\tau + c.$$

Definition 8. [16] The system (1), (2) has the *smooth dissipativity* property if there exist a $\mathcal{C}^1$ function $V : \mathcal{M} \to \mathbb{R}_+$, α_1, α_2, $\sigma \in \mathcal{K}_\infty$, a continuous positive definite function α_4, and a continuous output map $h : \mathcal{M} \to \mathbb{R}^p$ with

$$|x|_{\mathcal{W}} = 0 \;\Rightarrow\; h(x) = 0, \quad \forall x \in \mathcal{M}$$

such that (5) is satisfied for all $x \in \mathcal{M}$ and the following dissipation inequality holds for all $(x, u) \in \mathcal{M} \times \mathcal{U}$:

$$DV(x)f(x, u) \leq -\alpha_4(|h(x)|) + \sigma(|u|). \tag{7}$$

Definition 9. [16] The system (1), (2) has the *weak zero-detectability* property if the following relation holds:

$$h(x(t, x_0; 0)) \equiv 0,\ \forall t \geq 0 \Rightarrow |x(t, x_0; 0)|_{\mathcal{W}} \to 0$$

as $t \to \infty$.

The principal results connecting these properties are as following:

Theorem 1. *[3] Consider a nonlinear system (1) and let $\mathcal{W}$ be as in Assumption 1. Then the following are equivalent:*

1) *The system enjoys the pAG or AG property.*
2) *The system admits an ISS Lyapunov function.*
3) *The system admits an ISS Lyapunov function constant on invariant sets.*
4) *The system admits a practical ISS Lyapunov function.*
5) *The system enjoys the LIM property and the pGS.*

The system as in (1) that satisfies these properties will be called ISS in the multistable sense *with respect to the set $\mathcal{W}$ and input u.*

Theorem 2. *[16] Consider a nonlinear system (1) and let $\mathcal{W}$ be as in Assumption 1. Then the following facts are equivalent:*

1) 0-*GATT and UBEBS properties.*
2) *Existence of an iISS Lyapunov function V such that $DV(x) = 0$ for all $x \in \mathcal{W}$.*
3) *Existence of an iISS Lyapunov function V.*
4) *Existence of an output function that makes the system smoothly dissipative and weakly zero-detectable.*

The system as in (1) that satisfies these properties will be called iISS in the multistable sense *with respect to the set $\mathcal{W}$ and the input u.*

C. Problem statement

In this paper, we deal with a subclass of nonlinear dynamical systems (1), (2) affine in the input of the following form:

$$\dot{x}(t) = f(x(t), v(t)) + G(x(t))u(t), \quad t \in \mathbb{R}_+, \tag{8}$$
$$y(t) = h(x(t)), \tag{9}$$

where $x(t) \in \mathcal{M}$ is the state vector, $v(t) \in \mathbb{R}^k$ is a disturbance, $u(t) \in \mathbf{U} \subseteq \mathbb{R}^m$ is the input vector, u is an element of $\mathcal{U}$, the set of admissible controls $\mathbb{R}_+ \to \mathbf{U}$ (locally essentially bounded and measurable signals), and $y(t) \in \mathcal{Y} \subseteq \mathbb{R}^p$ is the output vector. For this system $f : \mathcal{M} \times \mathbb{R}^k \to \mathbb{R}^n$ and the columns of the matrix function $G : \mathcal{M} \to \mathbb{R}^{n \times m}$ are assumed to be locally Lipschitz continuous on $\mathcal{M}$, $h : \mathcal{M} \to \mathbb{R}^p$ is continuously differentiable, $h(0) = 0$, and $f(0, 0) = 0$.

Systems in the form (8), (9) are said to be *ISS (iISS) stabilizable* if there exists a control law $u = K(x)$, with $K : \mathcal{M} \to \mathbf{U} \subseteq \mathbb{R}^m$, so that the closed-loop system has the ISS (iISS) property in $v \in \mathbb{R}^k$ with respect to $\mathcal{W}$. Therefore, the problem studied in this work can be formally determined as follows:

Problem. Consider the systems described by the affine nonlinear model (8), (9). Under Assumption 1, find conditions for these systems to be ISS (iISS) stabilizable.

III. Main results

As for ISS (iISS) stabilization with respect to a compact set [21], [28], we will look for existence conditions of a stabilizing feedback using the CLF framework. To present our main result, we introduce a suitable notion of ISS (iISS) CLF from the point of view of multistability framework.

Definition 10. A *practical ISS CLF* for the system (8), (9) and control $u(t) \in \mathbf{U} \subseteq \mathbb{R}^m$ is a differentiable function $V : \mathcal{M} \to \mathbb{R}_+$ satisfying:

1) There exist class $\mathcal{K}_\infty$ functions α_1 and α_2 such that (5) holds for all $x \in \mathcal{M}$.
2) There exist class $\mathcal{K}_\infty$ functions χ and α_3 and a constant $q \geq 0$ such that for all $x \in \mathcal{M}$ and all $v \in \mathbb{R}^k$

$$\inf_{u \in \mathcal{U}} \{a(x, v) + b(x)u\} \leq -\alpha_3(|x|_{\mathcal{W}}) + \chi(|v|) + q, \tag{10}$$

where $a(x, v) = DV(x)f(x, v)$, $b(x) = DV(x)G(x)$. If (10) holds for $q = 0$, then V is an *ISS CLF*. In addition, such V is an *iISS CLF* if the both items are satisfied for $q = 0$ and a continuous positive definite function α_3.

The inequality (10) can be rewritten as follows:

$$\inf_{u \in \mathcal{U}} \{a(x, v) - \tilde{\chi}(|v|) - q + b(x)u\} \leq -\alpha_3(|x|_{\mathcal{W}}),$$

where $\tilde{\chi} \in \mathcal{K}_\infty$ is such that $\tilde{\chi}(s) \geq \chi(s)$ for all $s \in \mathbb{R}_+$ and there exists a function

$$\psi(x) = \sup_{v \in \mathbb{R}^k} \{a(x, v) - \tilde{\chi}(|v|) - q\},$$

e.g., take $\tilde{\chi}(s) = \max\{\chi(s), \sup_{|x|,|v|\leq s}|a(x,v)|\}$ [21], then with such a choice $\psi(x) = \sup_{|v|\leq|x|}\{a(x,v) - \tilde{\chi}(|v|) - q\}$. By construction $\psi(x) \leq 0$ for all $x \in \mathcal{W}$, and since the function a is at least Lipschitz continuous in x, then the function ψ admits the same property. In addition, for all $x \in \mathcal{W}$

$$\inf_{u\in\mathcal{U}}\{\psi(x) + b(x)u\} \leq -\alpha_3(|x|_{\mathcal{W}}).$$

Finally, define

$$\Psi(x) = \psi(x) + \frac{1}{2}\alpha_3(|x|_{\mathcal{W}}),$$

then for all $x \in \mathcal{M}$ and $v \in \mathbb{R}^k$:

$$\Psi(x) \geq \frac{1}{2}\alpha_3(|x|_{\mathcal{W}}) + a(x,v) - \tilde{\chi}(|v|) - q,$$
$$\inf_{u\in\mathcal{U}}\{\Psi(x) + b(x)u\} \leq -\frac{1}{2}\alpha_3(|x|_{\mathcal{W}}).$$

Definition 11. A differentiable function $V : \mathcal{M} \to \mathbb{R}$ possesses a variant of the classical *small control property (SCP)* with respect to $x \in \mathcal{M}$ if it takes constant values on any $\mathcal{W}_i$ with $i = 1, \dots, k$, from the decomposition of $\mathcal{W}$, and for each $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x|_{\mathcal{W}} < \delta$ there exists some control $u \in \mathbf{U} \subseteq \mathbb{R}^m$ with $|u| < \varepsilon$ such that

$$\Psi(x) + b(x)u < 0. \tag{11}$$

Note that for the both above cases (ISS/iISS), the condition (10) can be replaced with another one:

$$b(x) \equiv 0 \Rightarrow \Psi(x) \leq -\frac{1}{2}\alpha_3(|x|_{\mathcal{W}}), \tag{12}$$

for all $x \in \mathcal{M}$, and α_3 is a class $\mathcal{K}_\infty$ function or a positive definite function for the cases of ISS or iISS, respectively. Therefore, the condition (12) formulates the main restriction to check for ISS (iISS) CLF in the system (8), (9).

Following the CLF framework from [6], [28], we will use the "universal" control formula, which in our case takes the form:

$$u = K(x) = \kappa(\Psi(x), |b(x)|^2)b(x)^T, \tag{13}$$

where

$$\kappa(s,r) = \begin{cases} -\frac{s+\sqrt{s^2+r^2}}{r} & \text{if } x \notin \mathcal{W} \\ 0 & \text{if } x \in \mathcal{W} \end{cases}.$$

Then, according with the above developments we propose the following theorem and lemma. We omit the proofs due to space limitations.

Theorem 3. *Let an affine system (8) admit Assumption 1. If the function $V : \mathcal{M} \to \mathbb{R}_+$ is an ISS (iISS) CLF, then the feedback law (13) is continuous in $\mathcal{M}\setminus\mathcal{W}$ and it provides the ISS (iISS) property in v with respect to $\mathcal{W}$ for (8). If such a CLF satisfies the SCP given in Definition 11, then the control (13) is continuous on $\mathcal{M}$.*

Note that $V(x) = 0$ for $x \in \mathcal{A}$ (it reaches its minimum), when $\mathcal{A} \subset \mathcal{W}$, then the control (13) will provide almost global or local attraction of the set $\mathcal{A}$ in the case $v = 0$.

Our further directions of research will apply the smooth dissipativity property for iISS stabilization of (8), (9), but here we will formulate the following consequence of our main result:

Lemma 1. *Consider an affine system (8) that satisfies Assumption 1. Then there exists a continuous control that provides the ISS (iISS) property in v with respect to $\mathcal{W}$ for* (8) *if and only if the system admits an ISS (iISS) CLF with SCP.*

IV. Examples

In this section we will illustrate peculiarities and features of the proposed CLF in the multistable framework on two academic examples.

A. Duffing oscillator

Consider the following variant of Duffing oscillator:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + u(t) + v_1(t), \\ \dot{x}_2(t) &= -x_2(t) + x_1(t)[x_1^2(t) - 1] + v_2(t), \end{aligned}$$

where $x(t) = (x_1(t)\ x_2(t))^\top \in \mathcal{M} = \mathbb{R}^2$, $u(t) \in \mathbb{R}$ and $v(t) = (v_1(t)\ v_2(t))^\top \in \mathbb{R}^2$ have the same sense as before. It is straightforward to check that for $u = 0$ and $v = 0$ the system has three equilibria, thus, $\mathcal{W} = \{(-1,0),(0,0),(1,0)\}$. Linearization shows that the origin is a locally asymptotically stable, and the two equilibria, $(-1,0)$ and $(1,0)$ are unstable foci (see Fig. 1).

Select $\mathcal{A} = \{(-1,0),(1,0)\}$ as the set that will be almost globally attractive in the closed-loop system. To this end, as a CLF candidate, choose

$$V(x) = \frac{1}{4}(x_1^2 - 1)^2 + \frac{1}{2}x_2^2,$$

whose derivative for the oscillator takes the usual form $\dot{V} = a(x,v) + b(x)u$, where

$$a(x,v) = 2x_1(x_1^2-1)x_2 - x_2^2 + x_1(x_1^2-1)v_1 + x_2v_2,$$
$$b(x) = x_1(x_1^2 - 1).$$

Select $\chi(s) = s^2$, then

$$a(x,v) - \chi(|v|) = \begin{pmatrix} x_1(x_1^2-1) \\ x_2 \\ v_1 \\ v_2 \end{pmatrix}^\top Q \begin{pmatrix} x_1(x_1^2-1) \\ x_2 \\ v_1 \\ v_2 \end{pmatrix}$$
$$-\frac{1}{4}x_2^2 + 2.3x_1^2(x_1^2-1)^2,$$

where

$$Q = \begin{pmatrix} -2.3 & 1 & 0.5 & 0 \\ 1 & -0.75 & 0 & 0.5 \\ 0.5 & 0 & -1 & 0 \\ 0 & 0.5 & 0 & -1 \end{pmatrix}$$

is a negative definite matrix. Hence, we can take

$$\begin{aligned} \psi(x) &= -\frac{1}{4}x_2^2 + 2.3x_1^2(x_1^2-1)^2 \\ &\geq a(x,v) - \chi(|v|). \end{aligned}$$

Next, let

$$|x|_{\mathcal{W}} = \sqrt{x_2^2 + x_1^2(x_1^2-1)^2}$$

and $\alpha_3(s) = \frac{1}{4}s^2$, then we obtain

$$\Psi(x) = \psi(x) + \frac{1}{2}\alpha_3\left(|x|_{\mathcal{W}}\right),$$

and it is easy to check that

$$b(x) = 0 \Rightarrow \Psi(x) \leq -\frac{1}{2}\alpha_3\left(|x|_{\mathcal{W}}\right).$$

Consequently, V is an ISS CLF for the original system and for any $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x|_{\mathcal{W}} < \delta$ there exists some control $|u| < \varepsilon$ such that

$$\Psi(x) + b(x)u < 0,$$

then SCP holds, and the control (13) should be continuous in $\mathbb{R}^2$.

The behavior of the controlled system is shown in Fig. 1. Note that with the applied control the two unstable foci become almost attractive in the closed-loop system. Observe that by starting the system from the same initial conditions the control action forces the trajectories to different attractors. To generate the plots the noise signals were chosen $v_1(t) = v_2(t) = 0.2\sin(10t)$.

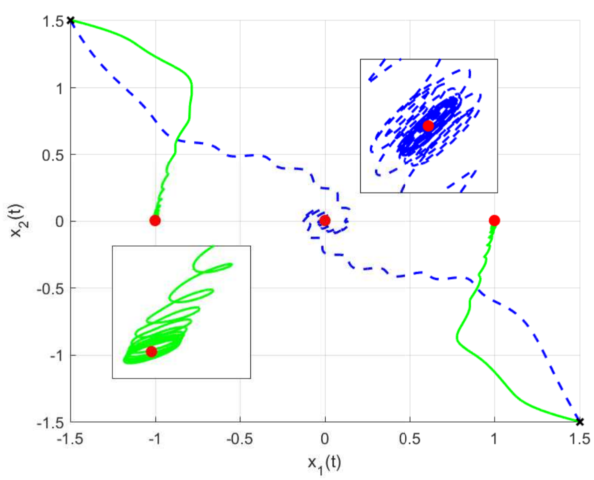

Figure 1: A trajectory convergence for the system with disturbance: without control (dashed line), with control (solid line)

B. Brockett oscillator

This system has the following model:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) + v_1(t),\\ \dot{x}_2(t) &= -x_1(t) - \beta x_2(t)\left(|x(t)|^2 - 1\right)\\ &\quad +\alpha u(t) + v_2(t),\end{aligned}$$

where again $x(t) = (x_1(t)\ x_2(t))^\top \in \mathcal{M} = \mathbb{R}^2$, $u(t) \in \mathbb{R}$ and $v(t) = (v_1(t)\ v_2(t))^\top \in \mathbb{R}^2$; $\alpha > 0$ and $\beta > 0$ are constant parameters. For $u = 0$ and $v = 0$ the system has the equilibrium at the origin and an attracting from almost all initial conditions limit cycle on the unit sphere $\mathbb{S} = \{x \in \mathbb{R}^2 : |x| = 1\}$, thus, $\mathcal{W} = \{(0,0), \mathbb{S}\}$ [1].

Assume that $\mathcal{A} = \{(0,0)\}$, which in our example will be locally attracting, and choose in this case as a CLF:

$$V(x) = \frac{1}{2}(x_1+x_2)^2 + \frac{1}{2}x_1^2,$$

whose derivative for the Brockett oscillator admits

$$\begin{aligned}a(x,v) &= -x_1^2 + (x_1+x_2)x_2\left(1+\beta-\beta|x|^2\right)\\ &\quad +(2x_1+x_2)v_1 + (x_1+x_2)v_2,\\ b(x) &= \alpha(x_1+x_2).\end{aligned}$$

Select $\chi(s) = s^2$, and similarly we get

$$a(x,v) - \chi(|v|) = \begin{pmatrix} x_1 \\ x_1+x_2 \\ v_1 \\ v_2 \end{pmatrix}^\top Q \begin{pmatrix} x_1 \\ x_1+x_2 \\ v_1 \\ v_2 \end{pmatrix}$$
$$-\frac{1}{2}x_1^2 + (x_1+x_2)^2 + (x_1+x_2)x_2[1+\beta-\beta|x|^2],$$

where

$$Q = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 \\ 0 & -1 & 0.5 & 0.5 \\ 0.5 & 0.5 & -1 & 0 \\ 0 & 0.5 & 0 & -1 \end{pmatrix}$$

is a negative definite matrix. Therefore, we can accept

$$\psi(x) = -\frac{1}{2}x_1^2 + (x_1+x_2)^2 + (x_1+x_2)x_2\left(1+\beta-\beta|x|^2\right),$$

and, finally take

$$|x|_{\mathcal{W}} = \frac{|x|}{\sqrt{1+|x|^2}}\sqrt{|1-|x|^2|}$$

with $\alpha_3(s) = \frac{1}{4}s^2$. Then, as before

$$\Psi(x) = \psi(x) + \frac{1}{2}\alpha_3\left(|x|_{\mathcal{W}}\right),$$

and the main restriction for V to be a CLF, *i.e.*

$$b(x) = 0 \Rightarrow \Psi(x) \leq -\frac{1}{2}\alpha_3\left(|x|_{\mathcal{W}}\right)$$

is satisfied and in fact, V is an ISS CLF for the original system. However, it is impossible to demonstrate the SCP property in this case due to the term $(x_1+x_2)x_2\left(1+\beta-\beta|x|^2\right)$ in $\Psi(x)$ which is not approaching to zero while x becomes close to $\mathbb{S}$. Therefore, in this example the control (13) will be discontinuous on $\mathbb{S}$.

The effect of the noise for both the open-loop and closed-loop systems with $\alpha = \beta = 1$ is shown in Fig. 2. Again, observe that by initiating the system from the same initial conditions the control action forces the trajectories to different attractors. In simulation the noise signals also were chosen as $v_1(t) = v_2(t) = 0.2\sin(10t)$.

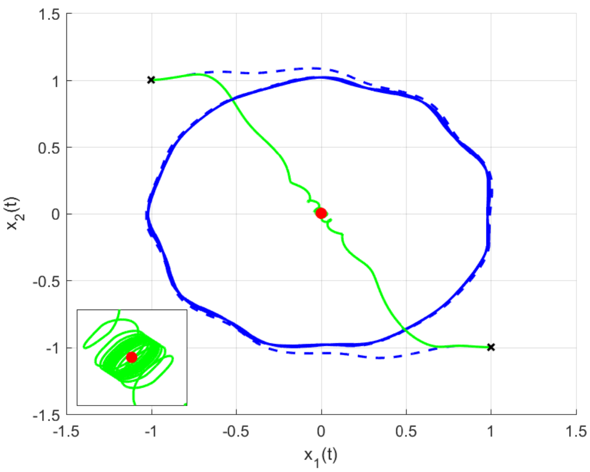

Figure 2: A trajectory convergence for the system with disturbance: without control (dashed line), with control (solid line)

V. Conclusion

In this work we dealt with a subclass of nonlinear multistable dynamical systems affine in the input. The main problem addressed here was to establish conditions for the existence of a feedback control that renders the ISS (iISS) stability property with respect to disturbances to the closed-loop system. In this direction, we have properly extended the notions of ISS (iISS) control Lyapunov functions and small control property within the multistability framework. In the multistable sense, the existence of a ISS (iISS) CLF satisfying the SCP implies the existence of a feedback control law that can be explicitly designed by the universal formula strategy. To exemplify the ISS (iISS) CLF application such a feedback control was designed to robustly stabilize two different nonlinear multistable systems.

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